Journal of Intelligent & Fuzzy Systems 28 (2015) 2471–2480
DOI:10.3233/IFS-141526
IOS Press

Knowledge reductions in generalized approximation space over two universes based on evidence theory

Weihua Xu[a,b,*], Wentao Li[a] and Shuqun Luo[a]
[a]*School of Mathematics and Statistics, Chongqing University of Technology, Chongqing, P.R. China*
[b]*Key Laboratory of Intelligent Perception and Systems for High-Dimensional Information, Ministry of Education, Nanjing University of Science and Technology, Nanjing, P.R. China*

Abstract. Knowledge reduction is one of the most important issues in rough set theory. According to various requirements and factors, information processing is based on two or more than two universes, other than single universe in many real-life cases. In this paper, we mainly investigate the knowledge reductions in generalized approximation space over two universes based on evidence theory. By defining the concepts of object belief and plausibility consistent sets over two universes, the object belief and plausibility consistent reductions are introduced in generalized approximation space over two universes. At the same time, the belief and plausibility significance reductions are also presented carefully in this space. Relationships among these proposed reductions are further studied, and it is proved that the object belief consistent reduction must be belief significance reduction and the object plausibility consistent reduction must be plausibility significance reduction.

Keywords: Approximation space, evidence theory, knowledge reduction, two universes

1. Introduction

Rough set theory proposed by Pawlak [9], is an extension of the classical set theory and could be regarded as a mathematical and soft computing tool to handle imprecision, vagueness and uncertainty in data analysis. This relatively new soft computing methodology has received great attention in recent years, and its effectiveness has been confirmed successful applications in many science and engineering fields, such as pattern recognition, data mining, image processing, medical diagnosis and so on [1, 3, 4]. Nowadays, the rough set theory has attracted more and more researchers. Due to the existence of uncertainty and complexity of particular problems, several extensions of the rough set model have been proposed in terms of various requirements, such as the variable precision rough set model [32, 35], rough set model based on tolerance relation [5], the Bayesian rough set model [17], the decision-theoretic rough set model [30], the fuzzy rough set model and the rough fuzzy set model [2] and others [12, 25].

One of the key issues of knowledge discovery is knowledge reduction. Usually, there are many objects and attributes in an information system. While some objects and attributes are not always needed based on lower and upper approximation. In representing knowledge, it is desirable to employ a minimum numbers of attributes without losing important information. A large variety of approaches have been proposed in the literature for effective and efficient reduction of knowledge [8, 11, 18, 34]. In rough set theory, we reduce redundant attributes and objects in the case of the classification unchanged.

Another important method used to deal with uncertain problems is the Dempster-Shafer theory of evidence. It was originated by Dempsters concept of lower and upper probabilities [1], and it is been extended by Shafer as a theory [13]. The basic representational structure is a belief structure in this theory which consists of a family of subsets, called focal

*Corresponding author. W.H. Xu, School of Mathematics and Statistics, Chongqing University of Technology, Chongqing, 400054, P.R. China. Tel.: +86 023 15998921583; E-mail: chxuwh@gmail.com.

elements, with associated individual positive weights summing to one. The primitive numeric measures induced by the belief structure are a dual pair of belief and plausibility functions. There are strong relationships between Dempster-Shafer theory of evidence and rough set theory. It has been checked that various belief structures are associated with various rough approximation spaces [16, 21–24, 31], satisfying the different dual pairs of lower and upper approximation operators derived from rough approximation spaces may be applied to illustrate the corresponding dual pairs of belief and plausibility functions induced by belief structures.

On the other hand, the expansion of single universe to two different universes is also very important nowadays. Yan et al. [26] studied on the model of rough set over dual-universe. Shen et al. [14] researched the variable precision rough set model over two universes and investigated the properties. Later on that, Yang et al. investigated the transformation of [27] bipolar fuzzy rough set model, the fuzzy probabilistic rough set based on two universities [29], and the bipolar fuzzy rough set model on two different universes [28]. For simplicity, more details about recent advancements of rough set model over two universes can be found in the literatures [6, 7, 10, 15, 19, 33].

This paper proposes knowledge reductions in generalized approximation space over two universes by combining evidence theory and rough set theory, and investigates the inner relationships of these reductions. The rest of this paper is organized as follows. Some preliminary concepts of rough set theory and evidence theory are introduced in Section 2. In the next section, we study evidence theory in generalized approximation space over two universes. In Section 4, we propose four types of reductions based on evidence theory. Furthermore, we investigate the connections of these reductions in Section 5. Finally, the paper is concluded by providing a summary and discussing the outlook for further research in Section 6.

2. Preliminaries

In this section,we will review some necessary definitions and concepts required in the sequel of this paper.

2.1. Pawlak rough sets

Let U be a finite and nonempty set called the universe, and R be an equivalence binary relation on U. The pair (U, R) is said to be a Pawlak approximation space. The equivalence relation R partitions U into disjoint subsets called equivalence classes. The elements in the same equivalence class are indiscernible. For any $X \subseteq U$, the lower and upper approximations are:

$$\underline{R}(X) = \{x | [x]_R \subseteq X\}; \overline{R}(X) = \{x | [x]_R \cap X \neq \emptyset\}.$$

The pair $(\underline{R}(X), \overline{R}(X))$ are the Pawlak rough set of X with respect to (U, R).

2.2. Rough theory based on two universes

Definition 2.1. [20] let R is an arbitrary binary relation from U to V. for any $x \in U$, $y \in V$, if xRy, i.e. $(x, y) \in R$, then x is called the predecessor of y and y is called the successor of x, denote:

$$R_s(x) = \{y \in V | xRy\},$$
$$R_p(y) = \{x \in U | xRy\}.$$

Definition 2.2. [20] Let (U, V, R) be a generalized approximation space, R is an arbitrary binary relation from U to V. For any $X \subseteq U$, $Y \subseteq V$, $x \in U$, $y \in V$ the lower and upper approximation of X, Y can be defined as follows:

$$\underline{R}_U(X) = \{y \in V | R_p(y) \subseteq X\},$$
$$\overline{R}_U(X) = \{y \in V | R_p(y) \cap X \neq \emptyset\};$$
$$\underline{R}_V(Y) = \{x \in U | R_s(x) \subseteq Y\},$$
$$\overline{R}_V(Y) = \{x \in U | R_s(x) \cap Y \neq \emptyset\}.$$

The pairs $(\underline{R}_U(X), \overline{R}_U(X))$and $(\underline{R}_V(Y), \overline{R}_V(Y))$ are the rough sets of X and Y in terms of generalized approximation space (U, V, R), respectively.

2.3. Evidence theory

In evidence theory, for a universe U, a *mass* function can be defined by a map $m : 2^U \to [0, 1]$, which is called a basic probability assignment and satisfies two axioms:

$$(1) \quad m(\emptyset) = 0$$
$$(2) \quad \sum_{X \subseteq U} m(X) = 1$$

If a subset $X \subseteq U, m(X) > 0$, then we say X is a focal element of m, (M, m) is called belief structure. Using the basic probability assignment, belief and plausibility functions of X in Pawlak approximation space are expressed as

$$Bel(X) = \sum_{Y \subseteq X} m(Y),$$

$$Pl(X) = \sum_{Y \cap X \neq \emptyset} m(Y).$$

From the above, we can know that *mass* function is a basic probability assignment in classical Pawlak approximation space.

3. Evidence theory in two universes approximation space

In this section, we will introduce evidence theory into generalized approximation space over two universes and discuss how to establish belief and plausibility functions. In generalized approximation space over two universes, R is neither a equivalence relation nor a general binary relation on the universe U, R is a general binary relation from U to V.

Definition 3.1. Let (U, V, R) be a generalized approximation space over two universes, R is an arbitrary binary relation from U to V, $R_p(y) = \{x \in U | xRy, y \in V\}$, $U/R = \{R_p(y) | y \in V\}$. For any $X \in U/R$, if denote $h(X) = \{y \in V | R_p(y) = X\}$, then we can define the mass function in (U, V, R) as follows:

$$m_1(X) = \frac{|h(X)|}{|V|}.$$

where mass function $m_1 : U/R \to [0, 1]$.

The value $m_1(X)$ represents the degree of belief that a specific element of U belongs to set X, but not to any particular subset of X. A subset $X \subseteq U$ satisfied $m_1(X) > 0$ is referred as to a focal element. We denote the family of all focal elements of m_1 by M_1. The pair (M_1, m_1) is called the first type of belief structure.

Definition 3.2. Let (U, V, R) be a generalized approximation space over two universes, R is an arbitrary binary relation from U to V, $R_s(x) = \{y \in V | xRy, x \in U\}$, $V/R = \{R_s(x) | x \in U\}$. For any $Y \in V/R$, if denote $k(Y) = \{x \in U | R_s(x) = Y\}$, then we can define the mass function in (U, V, R) as follows:

$$m_2(Y) = \frac{|k(Y)|}{|U|}.$$

where mass function $m_2 : V/R \to [0, 1]$.

The value $m_2(Y)$ represents the degree of belief that a specific element of V belongs to set Y, but not to any particular subset of Y. A subset $Y \subseteq V$ satisfied $m_2(Y) > 0$ is referred as to a focal element. We denote the family of all focal elements of m_2 by M_2. The pair (M_2, m_2) is called the second type of belief structure.

From the above definition, we can know that mass function m_1 in generalized approximation space over two universes satisfies two basic axioms. That is to say, for any $X \subseteq U$, $Y \subseteq V$ in (U, V, R), the following axioms hold.

$$(1) \qquad m_1(\emptyset) = 0$$

$$(2) \qquad \sum_{X \in U/R} m_1(X) = 1$$

The mass function m_2 also satisfies two basic axioms, for any $Y \subseteq V$, we have:

$$(1) \qquad m_2(\emptyset) = 0$$

$$(2) \qquad \sum_{Y \in V/R} m_2(Y) = 1$$

Associated with each belief structure in Pawlak approximation space based on classical equivalence relation, a pair of belief and plausibility functions can be derived.

Definition 3.3. Let (U, V, R) be a generalized approximation space over two universes. (M_1, m_1) is a belief structure in approximation space over two universes, $X \subseteq U$, $X^{'} \in U/R$, belief function $Bel_1 : 2^U \to [0, 1]$ and plausibility function $Pl_1 : 2^U \to [0, 1]$ can be defined as follows:

$$Bel_1(X) = \sum_{X' \subseteq X, X' \in U/R} m_1(X^{'});$$

$$Pl_1(X) = \sum_{X' \cap X \neq \emptyset, X' \in U/R} m_1(X^{'}).$$

In the same belief structure, belief function Bel_1 and plausibility function Pl_1 have duality property, that is to say,

$$Pl_1(X) = 1 - Bel_1(\sim X).$$

For any $X \in U/R$, $Bel_1(X) \leq Pl_1(X)$.

Definition 3.4. Let (U, V, R) be a generalized approximation space over two universes. (M_2, m_2) is a belief structure in approximation space over two universes,

$Y \subseteq V$, $Y' \in V/R$, belief function $Bel_2 : 2^V \to [0, 1]$ and plausibility function $Pl_2 : 2^V \to [0, 1]$ can be defined as follows:

$$Bel_2(Y) = \sum_{Y' \subseteq Y, Y' \in V/R} m_2(Y');$$

$$Pl_2(Y) = \sum_{Y' \cap Y \neq \emptyset, Y' \in V/R} m_2(Y').$$

In the same belief structure, belief function Bel_2 and plausibility function Pl_2 have duality property, that is to say,

$$Pl_2(Y) = 1 - Bel_2(\sim Y).$$

For any $Y \in V/R$, $Bel_2(Y) \leq Pl_2(Y)$.

Theorem 3.1. (See[31]) *Let (U, A, f) be an information system, for any $X \subseteq U$, $B \subseteq A$, denote*

$$Bel_B(X) = \frac{|\underline{R_B}(X)|}{|U|};$$

$$Pl_B(X) = \frac{|\overline{R_B}(X)|}{|U|},$$

then $Bel_B(X)$ is the belief function and $Pl_B(X)$ is the plausibility function of U. where the corresponding mass function is

$$m_B(Y) = \begin{cases} P(Y) & \text{if } Y \in U/R_B; \\ 0 & \text{otherwise.} \end{cases}$$

The detailed description of relationships between rough set theory and the Dempster-Shafer theory of evidence can be found in [16, 21, 22].

We can obtain the results in the following which present that the operators of the lower and the upper approximation in the generalized approximation (U, V, R) over two universes induce a pair of belief and plausibility functions respectively.

Theorem 3.2. *Let (U, V, R) be a generalized approximation space over two universes. $X \subseteq U$, denote*

$$Bel_U(X) = \frac{|\underline{R}_U(X)|}{|V|},$$

$$Pl_U(X) = \frac{|\overline{R}_U(X)|}{|V|},$$

then $Bel_U(X)$ and $Pl_U(X)$ are the belief function and plausibility function respectively and the corresponding mass function is

$$m_U(X) = \begin{cases} \frac{|h(X)|}{|V|} & \text{when } X \in U/R = \{R_p(y) | y \in V\} \\ 0 & \text{otherwise} \end{cases}$$

Proof. This theorem can be proved by Definition 3.1 and Definition 3.3. □

Theorem 3.3. *Let (U, V, R) be a generalized approximation space over two universes. $Y \subseteq V$, denote*

$$Bel_V(Y) = \frac{|\underline{R}_V(Y)|}{|U|},$$

$$Pl_V(Y) = \frac{|\overline{R}_V(Y)|}{|U|},$$

then $Bel_V(Y)$ and $Pl_V(Y)$ are the belief function and plausibility function respectively and the corresponding mass function is

$$m_V(Y) = \begin{cases} \frac{|k(Y)|}{|U|} & \text{when } Y \in V/R = \{R_s(x) | x \in U\} \\ 0 & \text{otherwise} \end{cases}$$

Proof. This theorem can be proved by Definition 3.2 and Definition 3.4.

Theorem 3.4. *Let (U, V, R) be a generalized approximation space over two universes. R, S are two general binary relations from U to V, $R \subseteq S$, for any $X \subseteq U$, $Y \subseteq V$, the following properties are hold.*

- *(1) $Bel_U^S(X) \leq Bel_U^R(X) \leq Pl_U^R(X) \leq Pl_U^S(X)$;*
- *(2) $Bel_V^S(Y) \leq Bel_V^R(Y) \leq Pl_V^R(Y) \leq Pl_V^S(Y)$.*

Proof. (1)Because $R \subseteq S$, so we have

$$\underline{S}_U(X) \subseteq \underline{R}_U(X) \text{ and } \overline{R}_U(X) \subseteq \overline{S}_U(X)$$
$$\Rightarrow |\underline{S}_U(X)| \leq |\underline{R}_U(X)| \text{ and } |\overline{R}_U(X)| \leq |\overline{S}_U(X)|$$
$$\Rightarrow \frac{|\underline{S}_U(X)|}{|V|} \leq \frac{|\underline{R}_U(X)|}{|V|} \text{ and } \frac{|\overline{R}_U(X)|}{|V|} \leq \frac{|\overline{S}_U(X)|}{|V|}$$
$$\Rightarrow Bel_U^S(X) \leq Bel_U^R(X) \text{ and } Pl_U^R(X) \leq Pl_U^S(X).$$

On the other hand, for any $X \subseteq U$, we have

$$\underline{R}_U(X) \subseteq \overline{R}_U(X) \Rightarrow Bel_U^R(X) \leq Pl_U^R(X).$$

This item is completed.

(2) This item can be proved similarly. □

Example 3.1. Given two generalized approximation spaces over two universes in Table 1 and Table 2.

From the above table, we have:

$$R_p(y_1) = \{x_2, x_6, x_9\}; \quad R_p(y_2) = \{x_1, x_3, x_6, x_8\};$$
$$R_p(y_3) = \{x_2, x_4, x_5, x_9\}; \quad R_p(y_4) = \{x_1, x_5, x_9, x_{10}\};$$
$$R_p(y_5) = \{x_3, x_4, x_7\}; \quad R_p(y_6) = \{x_3, x_5, x_8\};$$
$$R_p(y_7) = \{x_4, x_7\}; \quad R_p(y_8) = \{x_1, x_2, x_5, x_8, x_{10}\};$$
$$R_p(y_9) = \{x_2, x_4, x_5, x_7, x_8\}; \quad R_p(y_{10}) = \{x_1, x_6, x_9\}.$$

From the above table, we have:

$$S_p(y_1) = \{x_2, x_4, x_6, x_9, x_{10}\}; \quad S_p(y_2) = \{x_1, x_3, x_5, x_6, x_8, x_9\};$$
$$S_p(y_3) = \{x_2, x_4, x_5, x_7, x_9\}; \quad S_p(y_4) = \{x_1, x_2, x_3, x_5, x_7, x_9, x_{10}\};$$
$$S_p(y_5) = \{x_1, x_3, x_4, x_7, x_9\}; \quad S_p(y_6) = \{x_1, x_3, x_5, x_8, x_9\};$$
$$S_p(y_7) = \{x_2, x_4, x_7\}; \quad S_p(y_8) = \{x_1, x_2, x_5, x_7, x_8, x_{10}\};$$
$$S_p(y_9) = \{x_2, x_4, x_5, x_7, x_8, x_{10}\}; \quad S_p(y_{10}) = \{x_1, x_3, x_6, x_9\}.$$

From the above two tables, we have $R \subseteq S$. Let $X = \{x_1, x_2, x_3, x_5, x_7, x_8, x_{10}\}$, then we can easily obtain the lower and upper approximations of X by using the Definition 2.2

$$\underline{R}_U(X) = \{y_6, y_8\}, \quad \overline{R}_U(X) = V;$$
$$\underline{S}_U(X) = \{y_8\}, \quad \overline{S}_U(X) = V.$$

Then, we can gain the belief function and plausibility function as follows:

$$Bel_U^R(X) = \frac{|\underline{R}_U(X)|}{|V|} = \frac{1}{5}, \quad Pl_U^R(X) = \frac{|\overline{R}_U(X)|}{|V|} = 1;$$
$$Bel_U^S(X) = \frac{|\underline{S}_U(X)|}{|V|} = \frac{1}{10}, \quad Pl_U^S(X) = \frac{|\overline{S}_U(X)|}{|V|} = 1.$$

So, the $Bel_U^S(X) \leq Bel_U^R(X) \leq Pl_U^R(X) \leq Pl_U^S(X)$ holds.

4. The reductions of generalized approximation space over two universes

Reduction in rough sets theory is very important, because there are many redundant knowledge in information system or approximation space. In this section, we mainly introduce four types of the methods of reductions in generalized approximation space over two universes. The first method is the object belief consistent reduction; the second method is the object plausibility consistent reduction; the third method is the belief significance reduction; the last method is the plausibility significance reduction. In the following, we will investigate these reductions one by one.

4.1. The object belief consistent reduction

Definition 4.1. Let (U, V, R) be a generalized approximation space. R is a general binary relation from U to V, $U/R = \{R_p(y_1), R_p(y_2), \cdots, R_p(y_n)\}$, for any $X \in U/R$, $U' \subseteq U$ satisfy $Bel_{U'}^R(X) = Bel_U^R(X)$, then the U' is the object belief consistent set. If for any $U'' \subset U'$ is not the object belief consistent set. Then we call U' is a object belief consistent reduction.

Theorem 4.1. *For any generalized approximation space (U, V, R) over two universes, there must exist a object belief consistent reduction.*

Proof. For any $x_i \in U (i = 1, 2, \cdots, m)$, $Bel_{U-\{x_i\}}^R(X) \neq Bel_U^R(X)$, then the universe U is the object belief consistent reduction.

If exist $x_i \in U$, $Bel_{U-\{x_i\}}^R(X) = Bel_U^R(X)$ holds, then we need study $U' = U - \{x_i\}$. If for any $x_i' \in U'$,

Table 1.
A generalized approximation space (U, V, R)

(U, V, R)	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
x_1	0	1	0	1	0	0	0	1	0	1
x_2	1	0	1	0	0	0	0	1	1	0
x_3	0	1	0	0	1	1	0	0	0	0
x_4	0	0	1	0	1	0	1	0	1	0
x_5	0	0	1	1	0	1	0	1	1	0
x_6	1	1	0	0	0	0	0	0	0	1
x_7	0	0	0	0	1	0	1	0	1	0
x_8	0	1	0	0	0	1	0	1	1	0
x_9	1	0	1	1	0	0	0	0	0	1
x_{10}	0	0	0	1	0	0	0	1	0	0

Table 2.
A generalized approximation space (U, V, S)

(U, V, S)	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
x_1	0	1	0	1	1	1	0	1	0	1
x_2	1	0	1	1	0	0	1	1	1	0
x_3	0	1	0	1	1	1	0	0	0	1
x_4	1	0	1	0	1	0	1	0	1	0
x_5	0	1	1	1	0	1	0	1	1	0
x_6	1	1	0	0	0	0	0	0	0	1
x_7	0	0	1	1	1	0	1	1	1	0
x_8	0	1	0	0	0	1	0	1	1	0
x_9	1	1	1	1	1	1	0	0	0	1
x_{10}	1	0	0	1	0	0	0	1	1	0

$Bel^R_{U'-\{x_i'\}}(X) \neq Bel^R_U(X)$, then U' is the object belief consistent reduction; if exist $x_i' \in U'$, $Bel^R_{U'-\{x_i'\}}(X) = Bel^R_U(X)$ holds, then we need study $U'' = U' - \{x_i'\}$, repeat the above algorithm, we can obtain the object belief consistent reduction in the end.

Example 4.1. (Continue 3.1) Compute the reductions of the generalized approximation space (U, V, R) in terms of belief consistent set. We denote $U/R = \{y_1, y_2, \cdots, y_{10}\} = \{X_1, X_2, \cdots, X_{10}\}$. In the following, we can compute the lower and upper approximations of X_i as follows:

$$\underline{R}_U(X_1) = \{y_1\}, \qquad \underline{R}_U(X_2) = \{y_2\};$$
$$\underline{R}_U(X_3) = \{y_3\}, \qquad \underline{R}_U(X_4) = \{y_4\};$$
$$\underline{R}_U(X_5) = \{y_5, y_7\}, \qquad \underline{R}_U(X_6) = \{y_6\};$$
$$\underline{R}_U(X_7) = \{y_7\}, \qquad \underline{R}_U(X_8) = \{y_8\};$$
$$\underline{R}_U(X_9) = \{y_7, y_9\}, \qquad \underline{R}_U(X_{10}) = \{y_{10}\}.$$

Then, we can know:

$$Bel^R_U(X_1) = \frac{|\underline{R}_U(X_1)|}{|V|} = \frac{1}{10}, \qquad Bel^R_U(X_2) = \frac{|\underline{R}_U(X_2)|}{|V|} = \frac{1}{10};$$
$$Bel^R_U(X_3) = \frac{|\underline{R}_U(X_3)|}{|V|} = \frac{1}{10}, \qquad Bel^R_U(X_4) = \frac{|\underline{R}_U(X_2)|}{|V|} = \frac{1}{10};$$
$$Bel^R_U(X_5) = \frac{|\underline{R}_U(X_5)|}{|V|} = \frac{1}{5}, \qquad Bel^R_U(X_6) = \frac{|\underline{R}_U(X_6)|}{|V|} = \frac{1}{10};$$
$$Bel^R_U(X_7) = \frac{|\underline{R}_U(X_7)|}{|V|} = \frac{1}{10}, \qquad Bel^R_U(X_8) = \frac{|\underline{R}_U(X_8)|}{|V|} = \frac{1}{10};$$
$$Bel^R_U(X_9) = \frac{|\underline{R}_U(X_9)|}{|V|} = \frac{1}{5}, \qquad Bel^R_U(X_{10}) = \frac{|\underline{R}_U(X_{10})|}{|V|} = \frac{1}{10}.$$

In the following, we will remove the object x_i one by one firstly. We remove x_4, x_8, x_{10}, for any $X_i \in U/R$, the $\underline{R}_U(X_i)$ do not change, thus $Bel^R_{U-\{x_4,x_8,x_{10}\}}(X_i) = Bel^R_U(X_i)$ holds, and other objects can not be removed.

So the object belief consistent reduction of this approximation space is $\{x_1, x_2, x_3, x_5, x_6, x_7, x_9\}$.

4.2. The object plausibility consistent reduction

Definition 4.2. Let (U, V, R) be a generalized approximation space. R is a general binary relation from U to V, $U/R = \{R_p(y_1), R_p(y_2), \cdots, R_p(y_n)\}$, for any $X \in U/R$, $U^{'} \subseteq U$ satisfy $Pl^R_{U'}(X) = Pl^R_U(X)$, then the $U^{'}$ is the object plausibility consistent set. If for any $U^{''} \subset U^{'}$ is not the object plausibility consistent set. Then we call $U^{'}$ is a object plausibility consistent reduction.

Theorem 4.2. *For any generalized approximation space (U, V, R) over two universes, there must exist a object plausibility consistent reduction.*

Proof. This theorem can be proved similar to Theorem 4.1.

Example 4.2. (Continue 3.1) Compute the reductions of the generalized approximation space (U, V, R) in terms of plausibility consistent set. We denote $U/R = \{y_1, y_2, \cdots, y_{10}\} = \{X_1, X_2, \cdots, X_{10}\}$. In the following, we can compute the lower and upper approximations of X_i as follows:

In the following, remove x_7, x_{10}, for any $X_i \in U/R$, the $\overline{R}_U(X_i)$ do not change, thus $Pl^R_{U-\{x_7,x_{10}\}}(X_i) = Pl^R_U(X_i)$. By computing, we know that other objects can not be removed, so the object plausibility consistent reduction is $\{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9\}$.

4.3. The belief significance reduction

In rough set theory, we also can investigate reductions in the point of view of significance of object, that is to say, we remove the objects which are not significant or remove the objects which are less significant. In this subsection, we mainly study the reductions in the generalized approximation (U, V, R) over two universes in terms of belief significance of the objects. Our idea about reduction is that convert the values of belief significance into [0,1]. Then we save the objects which belief significance are more than 0 and remove the objects which belief significance equals to 0. So the objects set which element's belief significance are more than 0 is reduction in the generalized approximation space (U, V, R) over two universes.

Definition 4.3. Let (U, V, R) be a generalized approximation space. R is a general binary relation from U

$$\overline{R}_U(X_1) = \{y_1, y_2, y_3, y_4, y_8, y_9, y_{10}\}, \qquad \overline{R}_U(X_2) = \{y_1, y_2, y_4, y_5, y_6, y_8, y_9, y_{10}\};$$

$$\overline{R}_U(X_3) = \{y_1, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}, \qquad \overline{R}_U(X_4) = \{y_1, y_2, y_3, y_4, y_6, y_8, y_9, y_{10}\};$$

$$\overline{R}_U(X_5) = \{y_2, y_3, y_5, y_6, y_7, y_9\}, \qquad \overline{R}_U(X_6) = \{y_2, y_3, y_4, y_5, y_6, y_8, y_9\};$$

$$\overline{R}_U(X_7) = \{y_3, y_5, y_7, y_9\}, \qquad \overline{R}_U(X_8) = \{y_1, y_2, y_3, y_4, y_6, y_8, y_9, y_{10}\};$$

$$\overline{R}_U(X_9) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}, \qquad \overline{R}_U(X_{10}) = \{y_1, y_2, y_3, y_4, y_8, y_{10}\}.$$

Then, we can know:

$$Pl^R_U(X_1) = \frac{|\overline{R}_U(X_1)|}{|V|} = \frac{7}{10}, \qquad Pl^R_U(X_2) = \frac{|\overline{R}_U(X_2)|}{|V|} = \frac{4}{5};$$

$$Pl^R_U(X_3) = \frac{|\overline{R}_U(X_3)|}{|V|} = \frac{9}{10}, \qquad Pl^R_U(X_4) = \frac{|\overline{R}_U(X_2)|}{|V|} = \frac{4}{5};$$

$$Pl^R_U(X_5) = \frac{|\overline{R}_U(X_5)|}{|V|} = \frac{3}{5}, \qquad Pl^R_U(X_6) = \frac{|\overline{R}_U(X_6)|}{|V|} = \frac{7}{10};$$

$$Pl^R_U(X_7) = \frac{|\overline{R}_U(X_7)|}{|V|} = \frac{2}{5}, \qquad Pl^R_U(X_8) = \frac{|\overline{R}_U(X_8)|}{|V|} = \frac{4}{5};$$

$$Pl^R_U(X_9) = \frac{|\overline{R}_U(X_9)|}{|V|} = \frac{9}{10}, \qquad Pl^R_U(X_{10}) = \frac{|\overline{R}_U(X_{10})|}{|V|} = \frac{3}{5}.$$

$$I_{Bel}(x_1)=\frac{|\frac{1}{10}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{5}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{5}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{10}|}{10}$$
$$=0.01;$$

$$I_{Bel}(x_2)=\frac{|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{5}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{5}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{5}|}{10}$$
$$=0.01;$$

$$I_{Bel}(x_3)=\frac{|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{5}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{5}|+|\frac{1}{5}-\frac{2}{5}|+|\frac{1}{10}-\frac{1}{10}|}{10}$$
$$=0.03;$$

$$I_{Bel}(x_4)=\frac{|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{5}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{5}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{10}|}{10}$$
$$=0;$$

$$I_{Bel}(x_5)=\frac{|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{5}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{10}-\frac{1}{10}|+|\frac{1}{5}-\frac{1}{5}|+|\frac{1}{10}-\frac{1}{10}|}{10}$$
$$=0.01;$$

to V, denote $U/R=\{R_p(y_1), R_p(y_2), \cdots, R_p(y_n)\}=\{X_1, X_2, \cdots, X_n\}$, for any $X_j \in$ U/R $(j=1,2,\cdots,n)$, $x_i \in U(i=1,2,\cdots,m)$, the belief significance of object x_i is

$$I_{Bel}(x_i)=\frac{\sum_{j=1}^{n}|Bel_U^R(X_j)-Bel_{U-\{x_i\}}^R(X_j)|}{|V|}.$$

From the definition 4.3, we can know that the belief significance of object x_i have boundary, i.e., $0\le I_{Bel}(x_i)\le 1$.

Definition 4.4. Let (U, V, R) be a generalized approximation space. R is a general binary relation from U to V, the belief significance reduction of the generalized approximation space (U, V, R) over two universes is the set which consist of all objects which belief significance are more than 0.

Theorem 4.3. *For any generalized approximation space (U, V, R) over two universes, there must exist a belief significance reduction.*

Proof. This theorem can be proved similar to Theorem 4.1.

Example 4.3. (Continue 3.1) Compute the reduction of the generalized approximation space (U, V, R) in terms of belief significance. We denote $U/R=\{y_1, y_2, \cdots, y_{10}\}=\{X_1, X_2, \cdots, X_{10}\}$. In the following, we can compute the belief significance of x_i as follows:

similarly, we can get $I_{Bel}(x_6)=0.01$; $I_{Bel}(x_7)=0.01$; $I_{Bel}(x_8)=0$; $I_{Bel}(x_9)=0.03$; $I_{Bel}(x_{10})=0$.

So the reduction of the generalized approximation space (U, V, R) over two universes in terms of belief significance is $\{x_1, x_2, x_3, x_5, x_6, x_7, x_9\}$.

4.4. The plausibility significance reduction

In this subsection, we mainly study the reduction based on the plausibility significance, we save the objects which plausibility significance are more than 0 and remove the objects which plausibility significance equal to 0. So the objects set which element's plausibility significance are more than 0 is reduction in the generalized approximation space (U, V, R) over two universes.

Definition 4.5. Let (U, V, R) be a generalized approximation space. R is a general binary relation from U to V, denote $U/R=\{R_p(y_1), R_p(y_2), \cdots, R_p(y_n)\}=\{X_1, X_2, \cdots, X_n\}$, for any $X_j \in U/R(j=1,2,\cdots,n)$, $x_i \in U(i=1,2,\cdots,m)$, the plausibility significance of object x_i is

$$I_{Pl}(x_i)=\frac{\sum_{j=1}^{n}|Pl_U^R(X_j)-Pl_{U-\{x_i\}}^R(X_j)|}{|V|}.$$

From the definition 4.3, we can know that the plausibility significance of object x_i have boundary, i.e., $0\le I_{Pl}(x_i)\le 1$.

Definition 4.4. Let (U, V, R) be a generalized approximation space. R is a general binary relation from U to V, the plausibility significance reduction of the generalized approximation space (U, V, R) over two universes is the set which consist of all objects which plausibility significance are more than 0.

Theorem 4.4. *For any generalized approximation space (U, V, R) over two universes, there must exist a plausibility significance reduction.*

Proof. This theorem can be proved similar to Theorem 4.1.

Example 4.4. (Continue 3.1) Compute the reductions of the generalized approximation space (U, V, R) in terms of plausibility significance. We denote $U/R = \{y_1, y_2, \cdots, y_{10}\} = \{X_1, X_2, \cdots, X_{10}\}$. In the following, we can compute the plausibility significance of x_i as follows:

$$I_{Pl}(x_1) = \frac{|\frac{7}{10} - \frac{7}{10}| + |\frac{8}{10} - \frac{7}{10}| + |\frac{9}{10} - \frac{9}{10}| + |\frac{8}{10} - \frac{7}{10}| + |\frac{6}{10} - \frac{6}{10}| + |\frac{7}{10} - \frac{7}{10}| + |\frac{4}{10} - \frac{4}{10}| + |\frac{8}{10} - \frac{7}{10}| + |\frac{9}{10} - \frac{9}{10}| + |\frac{6}{10} - \frac{5}{10}|}{10}$$

$$= 0.04;$$

$$I_{Pl}(x_2) = \frac{|\frac{7}{10} - \frac{6}{10}| + |\frac{8}{10} - \frac{8}{10}| + |\frac{9}{10} - \frac{9}{10}| + |\frac{8}{10} - \frac{8}{10}| + |\frac{6}{10} - \frac{6}{10}| + |\frac{7}{10} - \frac{7}{10}| + |\frac{4}{10} - \frac{4}{10}| + |\frac{8}{10} - \frac{7}{10}| + |\frac{9}{10} - \frac{8}{10}| + |\frac{6}{10} - \frac{6}{10}|}{10}$$

$$= 0.03;$$

$$I_{Pl}(x_3) = \frac{|\frac{7}{10} - \frac{7}{10}| + |\frac{8}{10} - \frac{7}{10}| + |\frac{9}{10} - \frac{9}{10}| + |\frac{8}{10} - \frac{8}{10}| + |\frac{6}{10} - \frac{4}{10}| + |\frac{7}{10} - \frac{6}{10}| + |\frac{4}{10} - \frac{4}{10}| + |\frac{8}{10} - \frac{8}{10}| + |\frac{9}{10} - \frac{9}{10}| + |\frac{6}{10} - \frac{6}{10}|}{10}$$

$$= 0.04;$$

$$I_{Pl}(x_4) = \frac{|\frac{7}{10} - \frac{7}{10}| + |\frac{8}{10} - \frac{8}{10}| + |\frac{9}{10} - \frac{7}{10}| + |\frac{8}{10} - \frac{8}{10}| + |\frac{6}{10} - \frac{5}{10}| + |\frac{7}{10} - \frac{7}{10}| + |\frac{4}{10} - \frac{3}{10}| + |\frac{8}{10} - \frac{8}{10}| + |\frac{9}{10} - \frac{9}{10}| + |\frac{6}{10} - \frac{6}{10}|}{10}$$

$$= 0.04;$$

$$I_{Pl}(x_5) = \frac{|\frac{7}{10} - \frac{7}{10}| + |\frac{8}{10} - \frac{8}{10}| + |\frac{9}{10} - \frac{8}{10}| + |\frac{8}{10} - \frac{6}{10}| + |\frac{6}{10} - \frac{6}{10}| + |\frac{7}{10} - \frac{5}{10}| + |\frac{4}{10} - \frac{4}{10}| + |\frac{8}{10} - \frac{8}{10}| + |\frac{9}{10} - \frac{9}{10}| + |\frac{6}{10} - \frac{6}{10}|}{10}$$

$$= 0.05;$$

similarly, we can get $I_{Pl}(x_6) = 0.02$; $I_{Pl}(x_7) = 0$; $I_{Pl}(x_8) = 0.02$; $I_{Pl}(x_9) = 0.04$; $I_{Pl}(x_{10}) = 0$.

So the reduction of the generalized approximation space (U, V, R) over two universes in terms of plausibility significance is $\{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9\}$.

In this section, we only discussed the reductions based on the universe U, and the reduction based on V in generalized approximation space (U, V, R) over two universes can be obtain similarly.

5. The relationships of these reductions

Theorem 5.1. An object belief consistent reduction is belief significance reduction, and belief significance reduction is object belief consistent reduction.

Proof. If $U^{'}$ is the object belief consistent reduction of the generalized approximation space (U, V, R) over two universes, then we have $Bel^R_{U'}(X) = Bel^R_U(X)$, so, for any $x_i \in U^{'}$, $X_j \in U/R$, so $|Bel^R_U(X_j) - Bel^R_{U-\{x_i\}}(X_j)| \geq 0$, hence we have $I_{Bel}(x_i) = \frac{\sum_{j=1}^{n} |Bel^R_U(X_j) - Bel^R_{U-\{x_i\}}(X_j)|}{|V|} \geq 0$. Consequently, the $U^{'}$ is the belief significance reduction.

On the other hand, if $U^{'}$ is the belief significance reduction of the generalized approximation space (U, V, R) over two universes, for any x_i, $I_{Bel}(x_i) = \frac{\sum_{j=1}^{n} |Bel^R_U(X_j) - Bel^R_{U-\{x_i\}}(X_j)|}{|V|} = 0$, then $|Bel^R_U(X_j) - Bel^R_{U-\{x_i\}}(X_j)| = 0$, so $Bel^R_{U-\{x_i\}}(X) = Bel^R_U(X)$, hence $Bel^R_{U'}(X) = Bel^R_U(X)$, where $U^{'}$ is the object set removed all objects which $I_{Bel}(x_i) = 0$

Theorem 5.2. *An object plausibility consistent reduction is plausibility significance reduction, and plausibility significance reduction is the object plausibility consistent reduction.*

Proof. According to Theorem 5.1, this theorem can be proved similarly.

6. Conclusions

In this paper, we combined the rough set theory and evidence theory to propose four types of reductions in generalized approximation space based on two universes. The first type of reduction is the object belief consistent reduction; the second type of reduction is the object plausibility consistent reduction the third type of reduction is the belief significance reductions; the last type of reduction is the plausibility significance reductions. Moreover, we studied the relationships of these reductions. There reductions are convenient for mining the useful knowledge in generalized approximation space.

Acknowledgements

This work is supported by Natural Science Foundation of China (No. 61105041), National Natural Science Foundation of CQ CSTC (No. cstc 2013jcyjA40051), and Key Laboratory of Intelligent Perception and Systems for High-Dimensional Information (Nanjing University of Science and Technology), Ministry of Education (No. 30920140122006).

References

[1] A.P. Dempster, Upper and lower probabilities induced by a multivalued mapping, *Annals of Mathematical Statistics* **38**(2) (1967), 325–339.
[2] D. Dübois and H. Prade, Rough fuzzy sets and fuzzy rough sets, *International Journal of General Systems* **17** (1990), 191–209.
[3] I. Düntsch and G. Gediga, Uncertainty measures of rough set prediction, *Artificial Intelligence* **106** (1998), 109–137.
[4] R. Jensen and Q. Shen, Fuzzy-rough sets assisted attribute selection, *IEEE Transactions on Fuzzy Systems* **15** (2007), 73–89.
[5] M. Kryzkiewicz, Rough set approach to incomplete information systems, *Information Sciences* **113** (1999), 271–292.
[6] T.J. Li and W.X. Zhang, Rough fuzzy approximations on two universes of discourse, *Information Sciences* **178**(3) (2008), 892–906.
[7] G.L. Liu, Rough set theory based on two universal sets and its applications, *Knowledge Based Systems* **23**(2) (2010), 110–115.
[8] J.S. Mi, W.Z. Wu and W.X. Zhang, Approaches to knowledge reduction based on variable precision rough sets model, *Information Sciences* **159**(3-4) (2004), 255–272.
[9] Z. Pawlak, Rough sets, *International Journal of Computer and Information Sciences* **11** (1982), 341–356.
[10] D.W. Pei and Z.B. Xu, Rough set models on two universes, *International Journal of General Systems* **33** (2004), 569–581.
[11] J.F. Peters and A. Skowron, A rough set approach to knowledge discovery, *International Journal of Intelligent System* **17**(2) (2002), 109–112.
[12] Y.H. Qian, H. Zhang, Y.L. Sang and J.Y. Liang, Multigranulation decision-theoretic rough sets, *International Journal of Approximation Reasoning* **55**(1) (2014), 225–237.
[13] G. Shafer, A mathematical theory of evidence. Princeton University Press, Princeton, 1976.
[14] Y.H. Shen and F.X. Wang, Variable precision rough set model over two universes and its properties, *Soft Computing* **15**(3) (2011), 557–567.
[15] L. Shu and X.Z. He, Rough set model with double universe of discourse, *IEEE International Conference on Information Reuse and Integration*, 2007, pp. 492–495.
[16] A. Skowron and J. Graymala-Busse, From rough set theory to evidence theory. In: R.R. Yager, M. Fedrizzi and J. Kacprzyk (eds), *Advance in the Dempster-Shafer theory of evidence*. Wiley, New York, 1994, pp. 193–236.
[17] D. Slezak and W. Ziarko, The investigation of the Bayesian rough set model, *International Journal of Approximate Reasoning* **40** (2005), 81–91.
[18] R. Slowinski, Rough set approach to decision analysis, *AI Expert* **10**(3) (1995), 19–25.
[19] B.Z. Sun and W.M. Ma, Fuzzy rough set model on two different universes and its application, *Applied Mathermatical Modeling* **35** (2011), 1798–1809.
[20] Y.P. Wang and W.Z. Wang, Constructive and algebraic methods of approximation operators over two universes, *Fuzzy Systems and Mathematics* **22** (2008), 151–154.
[21] W.Z. Wu, Y. Leung and W.X. Zhang, Connections between rough set theory and Dempster-Shafer theory of evidence, *International Journal of General Systems* **31**(4) (2002), 405–430.
[22] W.H. Xu, M.W. Shao and W.X. Zhang (2006) Knowledge reduction based on evidence reasoning theory in ordered information systems, *Lecture Notes in Computer Science* **4092**, 535–547.
[23] W.H. Xu, X.Y. Zhang, J.M. Zhong and W.X. Zhang (2010), Attribute reduction in ordered information systems based on evidence theory, *Knowledge and Information Systems* **25**(1), 169–184.
[24] W.H. Xu, W.X. Sun, Q.R. Wang and Y.F. Liu, Attribute reduction in tolerance information systems based on evidence theory, *Annals of Fuzzy Mathematics and Informatics* **4**(2) (2012), 267–279.
[25] W.H. Xu, W.X. Sun, X.Y. Zhang and W.X. Zhang, Multiple granulation rough set approach to ordered information systems, *International Journal of General Systems* **41**(5) (2012), 475–501.
[26] R.X. Yan, J.G. Zheng, J.L. Liu and Y.M. Zhai, Research on the model of rough set over dual-universes, *Knowledge-Based Systems* **23** (2010), 817–822.
[27] H.L. Yang, S.G. Li, Z.L. Guo and C.H. Ma, Transformation of bipolar fuzzy rough set model, *Knowledge-Based Systems* **27** (2012), 60–68.
[28] H.L. Yang, S.G. Li, S.Y. Wang and J. Wang, Bipolar fuzzy rough set model on two different universes and its application, *Knowledge-Based Systems* **35** (2012), 94–101.
[29] H.L. Yang, S.G. Li, S.Y. Wang and J. Wang, Fuzzy probabilistic rough set model on two universes and its applications, *International Journal of Approximate Reasoning* **54**(9) (2013), 1410–1420.
[30] Y.Y. Yao, Decision-theoretic rough set models. *In: Proceeding of Rough Sets and Knowledge Technology, Lecture Notes in Artificial Intelligence*, 4481, 2007, pp. 1–12.
[31] Y.Y. Yao and P. Lingras, Interpretations of belief functions in the theory of rough sets, *Information Sciences* **104** (1998), 81–106.
[32] X.Y. Zhang, Z.W. Mo, F. Xiong and W. Cheng, Comparative study of variable precision rough set model and graded rough set model, *International Journal of Approximate Reasoning* **53**(1) (2012), 104–116.
[33] H.Y. Zhang, W.X. Zhang and W.Z. Wu, Oncharacterization of the generalized interval-valued fuzzy rough sets on two universes of discourse, *International Journal of Approximate Reasoning* **51**(1) (2009), 56–70.
[34] W.X. Zhang, J.S. Mi and W.Z. Wu, Approaches to knowledge reductions in inconsistent systems, *International Journal of Intelligent Systems* **18** (2003), 989–1000.
[35] W. Ziarko, Variable precision rough set model, *Journal of Computer System Science* **46**(1) (1993), 39–59.